Control Decoupling Analysis for Gyroscopic Effects in Rolling Missiles

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The complex summation method is used to represent the pitch/yaw dynamic equations for a missile with autopilot. The particular forms of complex motion variables that are used enable cross coupling to be identified with imaginary coefficients in the resulting equations. A designer who wishes to produce uncoupled responses to commands for a maneuver then has a clearcut objective when choosing control laws, namely, to eliminate the imaginary coefficients from the response equations. This paper illustrates the use of this technique with an application to gyroscopic coupling resulting from a rolling airframe. It leads via transfer functions to the proposal of a zero-starred filter as an alternative to a pole/zero cancellation technique for eliminating the major source of coupling when the command is in a nonrolling frame of reference.

Nomenclature

f	= maneuver acceleration, m/s^2
G	= the parameter that expresses the amount of gyroscopic coupling, $P \cdot I_x/I_y$ (rad/s)
g	= gravitational acceleration, 9.81 m/s^2
I	= moment of inertia, kg · m^2
j	= $\sqrt{-1}$
M, N	= pitch and yaw moments, N · m
m	= missile mass, kg
P	= roll rate, rad/s
q, r	= pitch and yaw rates, rad/s
s	= Laplace operator
T	= time constant, s
t	= time, s
U	= forward speed, m/s
v, w	= components of airspeed caused by sideslip and pitch motion, m/s
x	= distance of the accelerometers forward of the center of mass, m
Y, Z	= yaw and pitch components of side force, N
δ	= $\delta_e + j\delta_r$, complex control surface angle, rad
δ_e, δ_r	= elevator and rudder angles, rad
μ	= $w + jv$, complex sideslip velocity, m/s
ϕ	= roll angle, rad
Ω	= $q + jr$, complex pitch/yaw angular rate, rad/s

Subscripts

D	= demanded value of
f	= relationship with the accelerometer
i	= component in inertial (command) axes, or input signal
o	= output signal
q, r	= result of pitch or yaw angular rates
v, w	= result of lateral or normal sideslip velocities
x	= value at x meters forward of the center of gravity, or component related to the x axis
y, z	= components related to the y or z axes
$\delta e, \delta r$	= result of elevator or rudder angle
δw	= abbreviation used as in $MZ_{\delta w} = M_\delta Z_w - M_w Z_\delta$
Ω	= relationship with the rate gyroscopes

Introduction

In the control of remotely piloted vehicles (RPV) and missiles it is frequently required that they respond to commands for maneuvers, in directions that are specified in inertial (nonrolling) axes, rather than the vehicles' (rolling) body axes. The maneuver commands may be expressed as components in orthogonal axes such as in vertical and horizontal directions.

If the vehicle is rolling with a low rate, then a simple resolution of the command signals through the roll angle will correct for the misalignment of the body's axes. However, if the roll rate is high, then cross couplings of significant magnitude will occur, which will not be corrected simply by resolving from the command to the vehicle's axes. A pure horizontal command, for instance, will produce a response that has an element of vertical in addition to the required horizontal motion. This will lead to a loss of performance[1,2] or even to instability in severe cases.

The designers of flight control systems (FCS) for missiles with high roll rates therefore will have to choose control laws that counteract the cross-coupling effects and produce responses that are decoupled in the axes of the command system. Horizontal maneuver commands should produce purely horizontal responses.

In the case of axisymmetric aircraft such as Cartesian missiles, a number of studies have been conducted[1,3–11] and the various sources of cross couplings classified.[1,4,12] Gyroscopic coupling has been identified as an important effect,[1,4] particularly when the roll inertia and/or the roll rate are high. Decoupling in the presence of just this phenomenon is the subject of this paper.

It is important when designing to decouple that the designer has a technique highlighting the presence of coupling in a way

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that will suggest how to counteract it. A standard way is to formulate the system equations in state-space terms, arranged so that vertical/horizontal cross couplings appear as elements off the leading diagonal of the matrices. The designer's objective is then to select control laws so as to cancel these terms. The formulation is that of a two-input two-ouput system.

Complex summation also can serve this purpose in the case of Cartesian missiles since they fall into the class of similar and antisymmetrically coupled (SAC) systems. After summation, the formulation appears as a single-input single-output (SISO) system with a wealth of analytical methods that may then be used, including state space, transfer functions, harmonic response, and so forth, albeit in generalized forms[15,21] that require a familiarization process before use.

Its use for decoupling purposes is demonstrated here. When followed by a state-space formulation, it will lead to the same design conclusion as if complex summation is not used. In this paper, the alternative route of using transfer functions has been followed, and this naturally leads to alternative solutions to the problem of decoupling.

In the case of gyroscopic coupling considered here, it leads to two prefilters. The preferred one is called a zero starred filter because it is derived from the complex conjugate for the system's zero. Whereas this is only valid for the particular type of FCS used for the purposes of illustration, nevertheless other types of FCS's have their equivalent types of prefilter.

The numerical cases make use of a standard missile, which is a tail-controlled Cartesian missile of uncertain parentage, the details of which are given in the Appendix.

Complex Summation

Complex summation refers to the combining of two equations or expressions P and Y after first multiplying one of them by $j(=\sqrt{-1})$. This is an old technique for use when P and Y are similar expressions such as when they refer to the pitch and yaw equations of an axisymmetric airframe. It was used in the 1950's for calculations leading to the design of FCS for Cartesian missiles, and a number of papers using the method appeared at that time.[13–20] When applied to the design of feedback systems, it calls for the generalization of the standard methods,[21,22] but these are not necessary for its use in the decoupling role that is put forward in this paper.

When P and Y are the dynamic equations for a Cartesian missile that is not rolling, then their complex sum can lead to equations in which all the coefficients are real. This does not happen automatically, but it can be made to happen by the appropriate choice of complex versions of the variables as in the following illustration.

The pitch and yaw equations for a nonrolling symmetric missile may be written as

Pitch P: $I_y\dot{q} - M_q q - M_w w =$ pitch control moment M (1a)
Yaw Y: $I_z\dot{r} - N_r r - N_v v =$ yaw control moment N (1b)

Complex summation to form $(P + jY)$ leads to

$$I_y(\dot{q} + j\dot{r}) - M_q(q + jr) - M_w(w - jv) = (M + jN) \quad (2)$$

Complex versions of the variables are introduced defined as: pitch/yaw angular rate $\Omega = q + jr$; and pitch/yaw sideslip velocity $\mu = w - jv$.

This enables Eq. (2) to be written as

$$I_y\dot{\Omega} - M_q\Omega - M_w\mu = (M + jN) \quad (3)$$

Note that Eq. (3) has real coefficients only as the result of the particular choice of complex versions of the variables, for example $(w - jv)$, as opposed to $(w + jv)$.

Note also that it has been assumed that $I_y = I_z$, $M_q = N_r$, and $M_w = -N_v$, which are caused by the similarity of pitch and yaw, both in their mass and their aerodynamic properties.

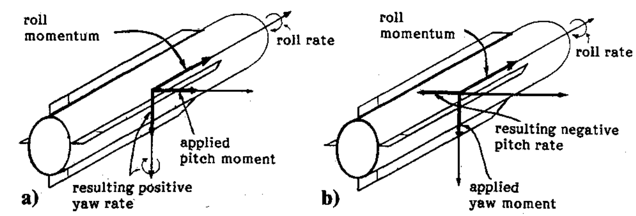

Fig. 1 Antisymmetric coupling showing a) pitch to yaw and b) yaw to pitch.

A similar preservation of real coefficients occurs for the translation

$$P \qquad m\dot{w} - Z_w w - mUq = Z \quad (4a)$$

$$Y \qquad m\dot{v} - Y_v v + mUr = Y \quad (4b)$$

Summing these as $(P - jY)$ leads to

$$m\dot{\mu} - Z_w\mu - mU\Omega = (Z - jY) \quad (5)$$

Aerodynamic similarity has led to the assumption $Y_v = Z_w$.

Now consider the case in which the airframe is rolling, with rate P. The equations may again be set up in nonrolling axes and become

$$P \qquad I_y\dot{q} + I_x Pr - M_q q - M_w w = M \quad (6a)$$

$$Y \qquad I_z\dot{r} - I_x Pq - N_r r - N_v v = N \quad (6b)$$

and by complex summation these become $(P + jY)$

$$I_y\dot{\Omega} - jI_x P\Omega - M_q\Omega - M_w\mu = (M + jN) \quad (7)$$

The term with the imaginary coefficient jI_x is associated with gyroscopic or inertial cross coupling between pitch and yaw.

The translational equations (4) and (5) are unchanged by the act of rolling, provided that the components all refer to nonrolling axes.

Gyroscopic coupling is one of several forms of coupling[1] that are of an antisymmetric nature. These forms have the characteristics that if a pitch motion produces a consequential positive yaw effect, then a similar yaw motion must produce a negative pitch effect in order that it shall be physically similar. Figure 1 illustrates this point by showing how an applied pitch moment will cause a positive yaw precession rate, whereas a positive yaw moment must cause a negative pitch precession rate in order to be physically similar.

Mathematically, the equations for SAC systems may be combined by complex summation, and complex versions of the variables found, such that antisymmetric cross couplings are identifiable with imaginary coefficients. Real coefficients will be associated with an absence of coupling between the two similar systems.

This means that the designer who wishes to achieve decoupling of the two similar channels has a clearcut objective, which is to produce response equations in which the coefficients are entirely real.

Another characteristic of complex summation is that although the order of SAC systems is twice the order of one of the component parts, the order of the equations after summing nevertheless remains the same as for just one of the parts. Thus, in the case of the missile airframe, the combination of pitch and yaw represents a fourth-order system [Eqs. (4) and (6)], and yet after summation, the designer is dealing with only a second-order set of equations [Eqs. (5) and (7)].

Furthermore, design work is done with a system representation that has no more inputs or outputs than one of the component parts. So, in the following example, there appears to be only one input, a demand for maneuver acceleration f_D, and one output, the achieved acceleration f_x. It may be dealt with as a SISO system.

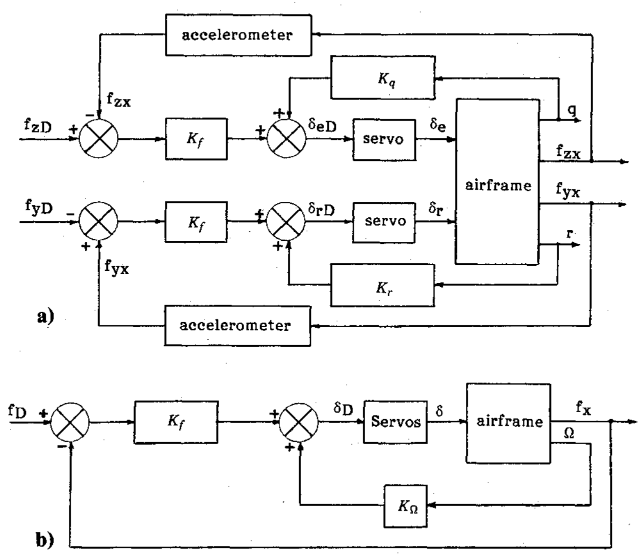

Fig. 2 Autopilot block diagram in a) complete form and b) as simplified by complex summation.

For the purpose of proceeding with gyroscopic decoupling, the airframe response equations may be summarized as in Eqs. (5) and (7), namely

$$m\dot{\mu} - Z_w\mu - mU\Omega = (Z - jY) \tag{8a}$$

$$I_y\dot{\Omega} - jI_xP\Omega - M_q\Omega - M_w\mu = (M + jN) \tag{8b}$$

To these may be added equations for the FCS.

It is assumed here that the autopilot is designed for a nonrolling airframe and that the designer wishes to modify it for use when the airframe is rolling at a constant roll rate P.

The baseline FCS is taken to be one that uses an accelerometer and rate gyroscope in both pitch and yaw channels. The block diagram is shown in Fig. 2a. However, because of complex summation, a simplified diagram may be used such as shown in Fig. 2b.

The rate gyroscopes provide information representing the angular rate components of Ω. Their outputs may be resolved if necessary in order to give components in nonrolling command axes.

The accelerometers measure the achieved acceleration components, and by means of resolution if necessary they provide measures of components in the command axes. If the instruments are placed at a distance x forward of the center of mass, then the components of the acceleration at this location, in the command axes, are

$$P \qquad f_{zx} = \dot{w} - x\dot{q} - Uq \tag{9a}$$

$$Y \qquad f_{yx} = \dot{v} + x\dot{r} + Ur \tag{9b}$$

And after complex summation, these become $(P - jY)$

$$f_x = \dot{\mu} - x\dot{\Omega} - U\Omega \tag{10}$$

Real coefficients are preserved by defining the complex version of the maneuver acceleration f_x as $f_{zx} - jf_{yx}$.

For convenience x will be taken to be equal to $Z_{\delta e}I_y/M_{\delta e}m$, which places the instruments at the center of percussion, a point of little significance beyond the fact that it simplifies the algebra, leading to a zero coefficient for s^2 in the denominator for f_x in Eq. (27).

The baseline FCS control laws are taken to be

$$P \qquad \delta_{eiD} = K_f(f_{zD} - f_{zx}) + K_q q \tag{11a}$$

$$Y \qquad \delta_{riD} = K_f(-f_{yD} + f_{yx}) + K_r r \tag{11b}$$

After complex summation, these may be written as $(P + jY)$

$$\delta_{iD} = K_f(f_D - f_x) + K_\Omega\Omega \tag{12}$$

where the complex version of the control surface deflections is

$$\delta_i \equiv \delta_{ei} + j\delta_{ri}$$

In numerical terms,

$$\delta_{iD} = 0.014(f_D - f_x) + 0.333\Omega \tag{13}$$

It should be noted that δ_{ei} and δ_{ri} are not the *actual* elevator and rudder deflections δ_e and δ_r, which are essentially relative to airframe axes. Equation (12) specifies the deflections that would be required if the control surfaces were always aligned with the nonrolling command axes. They are "fictitious deflections" and the actual surfaces will need to produce the same aerodynamic forces and moments on the airframe as are implied by Eq. (12).

From this, it follows that if

$$P \qquad \delta_e = \delta_{ei}\cos\phi + \delta_{ri}\sin\phi \tag{14a}$$

$$Y \qquad \delta_r = -\delta_{ei}\sin\phi + \delta_{ri}\cos\phi \tag{14b}$$

and via $(P + jY)$

$$\delta = \delta_i \exp(-j\phi) \tag{15}$$

then the control forces and moments to be inserted into Eqs. (8) are

$$Z - jY = Z_\delta\delta \qquad \text{and} \qquad M + jN = M_\delta\delta$$

It is assumed that $Z_\delta = Z_{\delta e} = -Y_{\delta r}$ and $M_\delta = M_{\delta e} = +N_{\delta r}$. The servos are assumed to respond very quickly so that to a first approximation

$$\delta_i = \delta_{iD} \tag{16}$$

In summary, after complex summation the system equations may be expressed as

Airframe:

$$m\dot{\mu} - Z_w\mu - mU\Omega = Z_\delta\delta_i \tag{17}$$

$$I_y\dot{\Omega} - jI_xP\Omega - M_q\Omega - M_w\mu = M_\delta\delta_i \tag{18}$$

Servo:

$$\delta_i = \delta_{iD} \tag{19}$$

Control law:

$$\delta_{iD} = K_f(f_D - f_x) + K_\Omega\Omega \tag{20}$$

Accelerometer:

$$f_x = \dot{\mu} - x\dot{\Omega} - U\Omega \tag{21}$$

These may be put into either state-space or transfer-function form. In state space, those elements of the matrices that represented cross coupling and appeared off the leading diagonal will now appear as complex elements, not necessarily off the leading diagonal.

For the standard missile on which the plotted results are based (see the Appendix), the numerical versions of these equations are

Airframe:

$$\dot{\mu} + \mu - 600\Omega = -60\delta_i \tag{22}$$

$$\dot{\Omega} + \left(0.8 - \frac{jPI_x}{I_y}\right)\Omega + 0.2\mu = -120\delta_i \tag{23}$$

Servo:

$$\delta_i = \delta_{iD} \tag{24}$$

Control law:

$$\delta_{iD} = 0.014(f_D - f_x) + 0.333\Omega \tag{25}$$

Accelerometer:

$$f_x = \dot{\mu} - 0.5\dot{\Omega} - 600\Omega \tag{26}$$

Gyroscopic Coupling

Gyroscopic cross coupling results from the fact that when a maneuver is required of a rolling aircraft, it is necessary to rotate the angular momentum vector. This will involve precession and nutation.

For the axisymmetric aircraft, these two behavior patterns may be forecast from the airframe dynamics, (17) and (18), and Eq. (21). If we assume the roll rate P is constant, the Laplace transforms lead to

$$f_x \Big/ \left[s\left(x\frac{MZ_{\delta w}}{I_y m} - \frac{Z_\delta M_q}{m I_y} + jP\frac{I_x Z_q}{I_y m} \right) + U\frac{MZ_{\delta w}}{I_y m} \right]$$
$$= \mu \Big/ \left[s\frac{Z_\delta}{m} + \left(U\frac{M_\delta}{I_y} - \frac{M_q Z_\delta}{I_y m} - jP\frac{I_x Z_\delta}{I_y m} \right) \right]$$
$$= \Omega \Big/ \left(s\frac{M_\delta}{I_y} - \frac{MZ_{\delta w}}{I_y m} \right)$$
$$= \delta_i \Big/ \left[s^2 - s\left(\frac{Z_w}{m} + \frac{M_q}{I_y} + jP\frac{I_x}{I_y} \right) + \left(-U\frac{M_w}{I_y} + \frac{Z_w M_q}{m I_y} + jP\frac{I_x Z_w}{I_y m} \right) \right] \tag{27}$$

where $MZ_{\delta w}$ is an abbreviation for $(M_\delta Z_w - M_w Z_\delta)$.

The numerical version of Eq. (27) is

$$f_x \Big/ \left[s\left(6 + j60P\frac{I_x}{I_y} \right) + 64{,}800 \right]$$
$$= \mu \Big/ \left[-60s - \left(72{,}000 - j60P\frac{I_x}{I_y} \right) \right]$$
$$= \Omega / (-120s - 108)$$
$$= \delta_i \Big/ \left[s^2 + s\left(1.8 - jP\frac{I_x}{I_y} \right) + \left(120.8 - jP\frac{I_x}{I_y} \right) \right] \tag{28}$$

The group PI_x/I_y appears to be a suitable measure of the amount of gyroscopic coupling and will be referred to here as G. It is, in fact, the frequency of the nutation mode, as may be seen from the airframe's characteristic equation (29) when the aerodynamic terms are omitted

$$s^2 - s\left(\frac{Z_w}{m} + \frac{M_q}{I_y} + jG \right) + \left(-U\frac{M_w}{I_y} + \frac{Z_w M_q}{m I_y} + jG\frac{Z_w}{m} \right) = 0 \tag{29}$$

or in numerical terms

$$s^2 + s(1.8 - jG) + (120.8 - jG) = 0 \tag{30}$$

The nutation mode clearly merges with and modifies the weathercock mode rather than retaining a separate identity. For a missile, the inertia ratio I_x/I_y is typically of order 0.05,

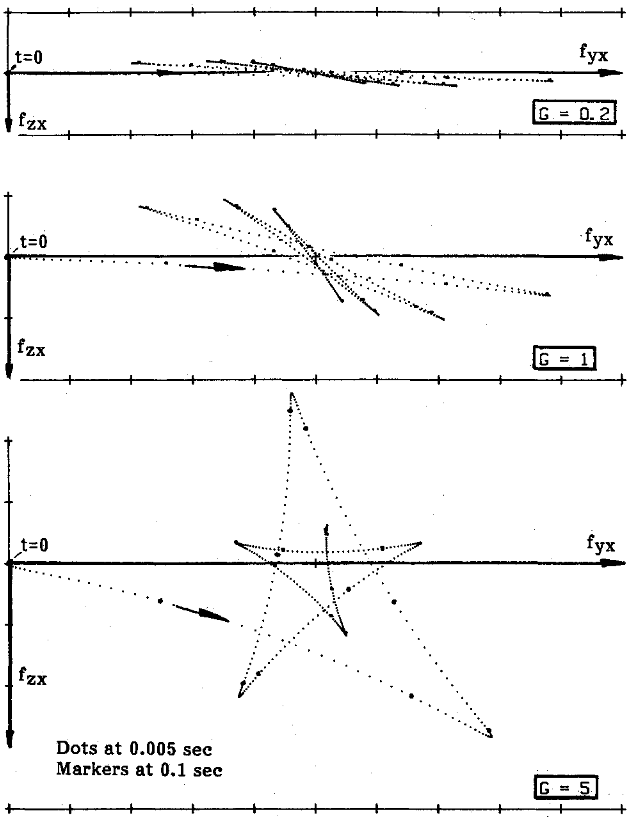

Fig. 3 Uncontrolled airframe response to a step change of "effective rudder."

and so, under normal roll rates, the frequency of its unadulterated nutation mode could be expected to be somewhat less than the weathercock frequency. Stated in another way, the imaginary terms in Eq. (29) can be expected to be small compared with the real ones, as is the case in Eq. (30), and it may be assumed that the weathercock mode will dominate the free behavior of a missile.

This is supported in Fig. 3, which shows the motion that follows a step change in δ_{ri}, for a range of values of G. The graphs show plots of horizontal vs vertical components of acceleration, the vertical ones being the undesirable out-of-plane components. The passage of time is represented by progression along the trajectories. When the roll rate is zero ($G = 0$), the motion is entirely in-plane and the behavior is that of the pure weathercock mode. As the gyroscopic effect increases, the out-of-plane motion gradually becomes apparent, but it is not a large effect until the roll rate is very large indeed. For a roll/pitch inertia ratio of 0.05, $G = 1$ corresponds to a roll rate of 20 rad/s or about 200 rpm.

In general, inertia ratios lie between 0 and 2 as the shape changes from infinitely slim to a disk, the flying saucer.

The effect of precession can be seen by applying the final value theorem to Eq. (27). These show, for example, that the final, constant pitch/yaw angular rate Ω and the maneuver acceleration f_x that will be caused by a constant control deflection δ_i obey the following:

$$\frac{f_x}{U(MZ_{\delta w}/I_y m)} = \frac{\Omega}{-(MZ_{\delta w}/I_y m)}$$
$$= \frac{\delta_i}{[-U(M_w/I_y) + (Z_w M_q/m I_y)] + jG(Z_w/m)} \tag{31}$$

Numerically,

$$\frac{f_x}{64{,}800} = \frac{\Omega}{-108} = \frac{\delta_i}{120.8 - jG} \tag{32}$$

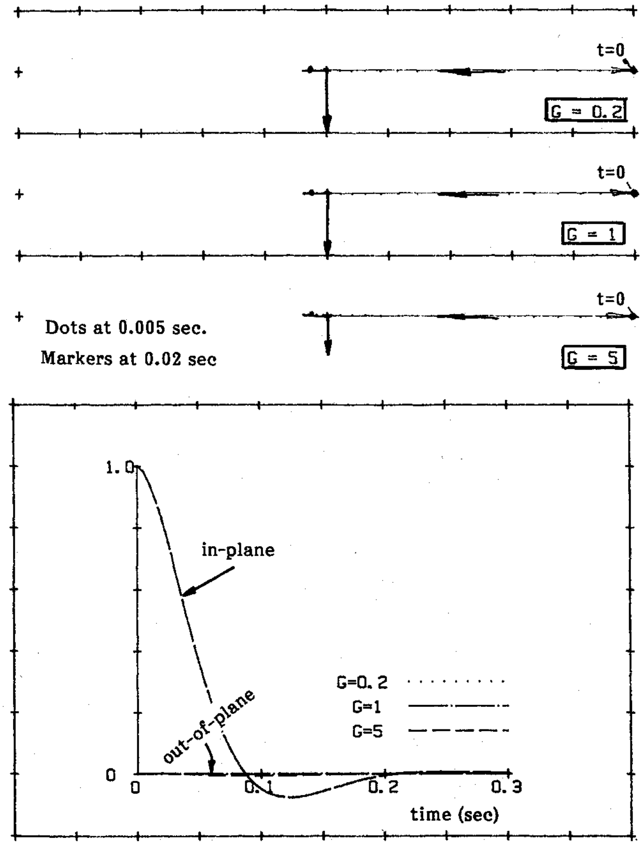

Fig. 4 Airframe with standard FCS: Response to an initial condition.

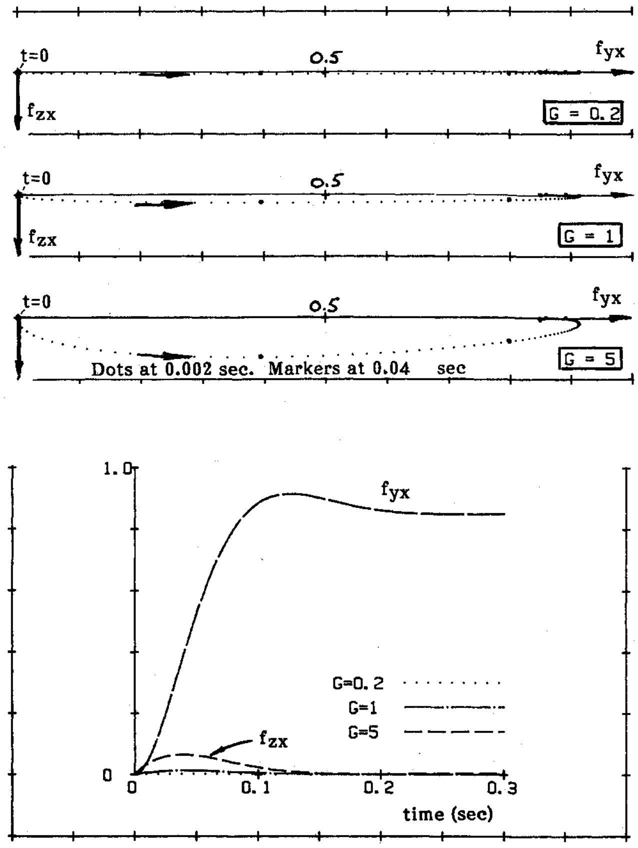

Fig. 5 Airframe with standard FCS: Response to a horizontal maneuver demand f_{yd}.

The imaginary term in these expressions is caused by precession, and it is small compared with the real term except at very large roll rates. This is demonstrated in Fig. 3 for the standard missile.

A byproduct of the complex summation method is that plots of orthogonal components of a variable, such as horizontal vs vertical components or components in body axes, are the plots of their real vs imaginary components. These relate closely to the theory and also to the physical behavior. Thus, the final value of f_x toward which the traces tend is proportional to the complex number fx/δ_i obtained from Eq. (32). The out-of-plane (imaginary) component is clearly smaller than the in-plane (real part), even at the highest value of G shown.

In summary, gyroscopic coupling affects both the modes and the response. For study purposes these may be treated separately. The magnitude of these effects may be forecast by studying the magnitude of the imaginary terms in the system equations, by comparison with the real terms.

Effect of the Flight Control System

If a FCS is fitted to the airframe, then the effect may be forecast from the complex equations (27) when they are modified to include the control law (20).

The FCS will modify the system equations (27); the ratios expressed there will now become equal to

$$K_f f_D \Big/ \left(s^2 + s\left\{ \left(-\frac{Z_w}{m} - \frac{M_q}{I_y} - jG \right) + K_f \left[x\frac{MZ_{\delta w}}{I_y m} - \frac{Z_\delta}{m} \times \left(\frac{M_q}{I_y} + jG \right) \right] - K_\Omega \frac{M_\delta}{I_y} \right\} + \left[\left(-U\frac{M_w}{I_y} + \frac{Z_w M_q}{m I_y} + jG\frac{Z_w}{m} \right) + K_f U \frac{MZ_{\delta w}}{I_y m} + K_\Omega \frac{MZ_{\delta w}}{I_y m} \right] \right) \tag{33}$$

Numerically, when $\delta_i = 0.014\ (f_D - f_x) + 0.333\Omega$, then the ratios (28) will also be equal to

$$\frac{0.014 f_D}{[s^2 + s(41.6 - j0.16G) + (1064 - jG)]} \tag{34}$$

This denominator, equated to zero, is the new characteristic equation; it may be compared with the airframe equation (29) or (30). The modified modal behavior may be seen in Fig. 4. This shows that gyroscopic coupling has a negligible effect, even at very high roll rates.

There is more evidence of a cross-coupling effect in the responses to a step change in the demand f_D; these are shown in Fig. 5. It follows that the cross coupling is mainly the result of a zero of the transfer function for f_x/f_D rather than a pole. However, their final value shows little cross coupling, indicating that the cross-plane response is not due to the precession that occurs in the final maneuver.

The numerator of the transfer function for f_x/f_D is

$$s\left[x\frac{MZ_{\delta w}}{I_y m} - \frac{Z_\delta}{m}\left(\frac{M_q}{I_y} + jG \right) \right] + U\frac{MZ_{\delta w}}{I_y m} \tag{35}$$

In numerical terms, the transfer function for fx/f_D becomes

$$\frac{f_x}{f_D} = \frac{s0.084(1 - j10G) + 907.2}{s^2 + s(41.6 - j0.16G) + (1064 - jG)} \tag{36}$$

This can be seen to forecast the results in Figs. 4 and 5.

1) The imaginary terms in the denominator are swamped by the real terms, indicating that gyroscopic coupling has not significantly affected the modal behavior of the "missile with autopilot"; this is dominated by the modified weathercock mode.

2) Precession does not have a major effect on the settling value when the missile reaches its final steady-state maneuver.

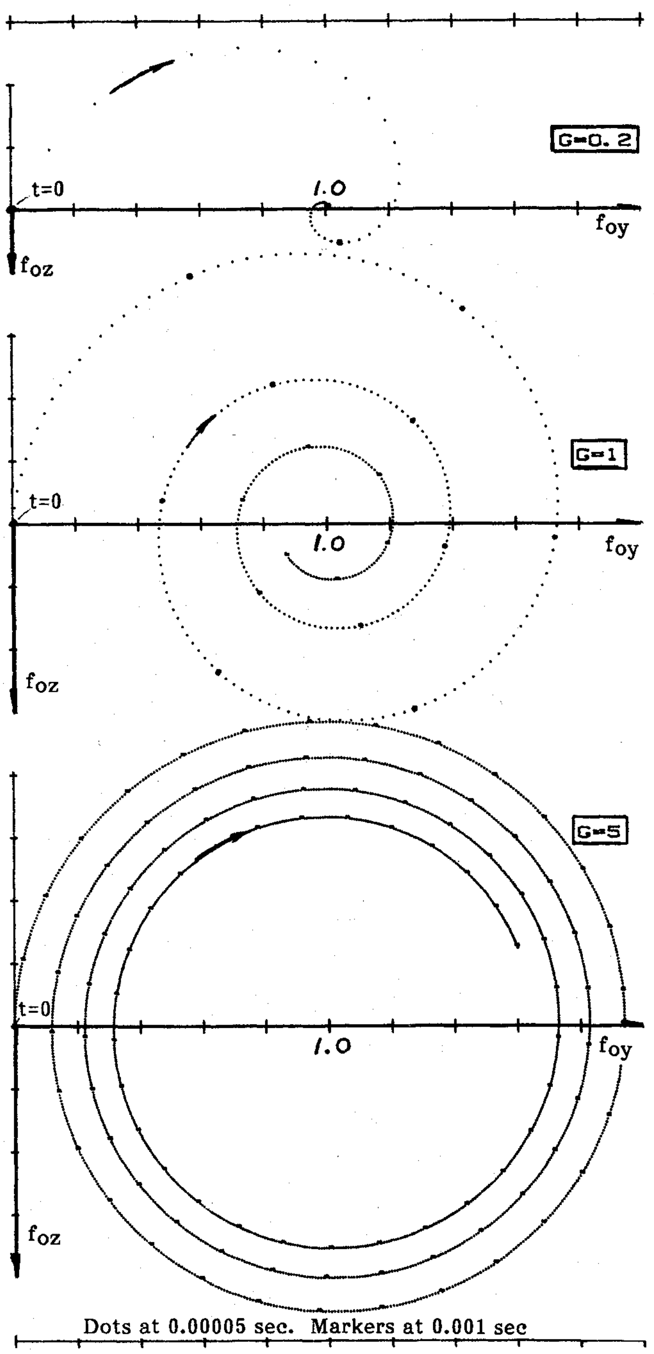

Fig. 6 Response of pole-zero cancellation filter (*PZ*) to a step demand f_{yd}.

The final value theorem indicates that the terminal value of f_x in response to a constant value of maneuver demand is given by

$$\frac{f_x}{f_D} = \frac{0.86}{1 - jG*10^{-3}} \tag{37}$$

3) The major effect of gyroscopic coupling will be due to the imaginary part of the coefficient of s in the numerator of the transfer function (36).

Decoupling Techniques

In the complex summation formulation that has been adopted here, there will be no cross coupling when all the coefficients in a transfer function are real. It is therefore necessary to modify the control laws of the FCS in order to achieve this. In particular, it is the maneuver g response that is of concern.

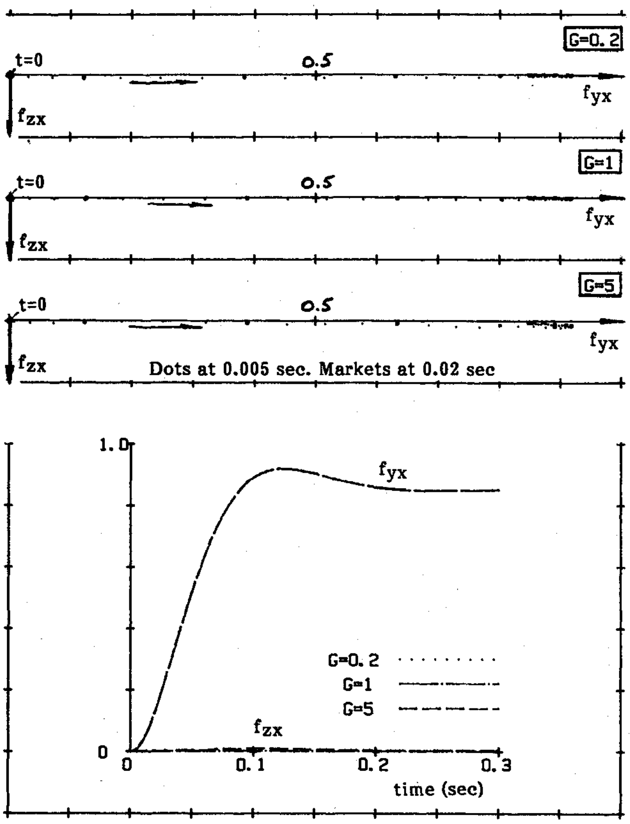

Fig. 7 Missile response with *PZ* filter fitted.

From the discussion here, it is clear that the complex coefficient that has the greatest influence is the coefficient of s in the numerator of Eq. (36). This produces a complex zero. Two methods of decoupling seem possible.

1) *Pole-zero cancellation* could be implemented by inserting a filter $F(s)$ that is the inverse of the numerator of Eq. (36) as a prefilter on the demand signal f_D in Fig. 2b. This filter has the transfer function

$$F(s) = 1/[1 + s0.9*10^{-4}(1 - j10G)] \tag{38}$$

It will introduce an unobservable mode, however.

The location of its pole indicates that it will be a stable oscillatory filter with a damping ratio equal to $1/\sqrt{1 + 100G^2})$, which may become unacceptably low if G exceeds about 0.3. Its undamped natural frequency will be $10^4/[0.9\sqrt{1 + 100G^2})]$ rad/s.

The response of this filter to a step demand for maneuver f_{yD} is shown in Fig. 6; this shows the oscillatory nature of the unobservable mode that it introduces.

The maneuver response of the missile is, however, largely decoupled. In Fig. 7, it may be seen that there is only an insignificant amount of vertical maneuver f_{zx} in response to the horizontal demand, even at the very high roll rate implied by $G = 5$. This figure also shows the time responses superimposed and the insensitivity to G is clear.

2) *A zero-starred filter* is again a prefilter operating on the demand f_D in Fig. 2b. It achieves real coefficients by the technique of multiplying complex conjugates.

When this technique is applied to the numerator of the transfer function (36), it suggests that a filter of form

$$F(s) = 1 + s0.9*10^{-4}(1 + j10G) \tag{39}$$

will lead to a response that is largely decoupled.

Although this introduces no unobservable oscillatory mode, it is a differentiating type of filter and it will need a low-pass filter in addition in order to limit the high frequency gain. The

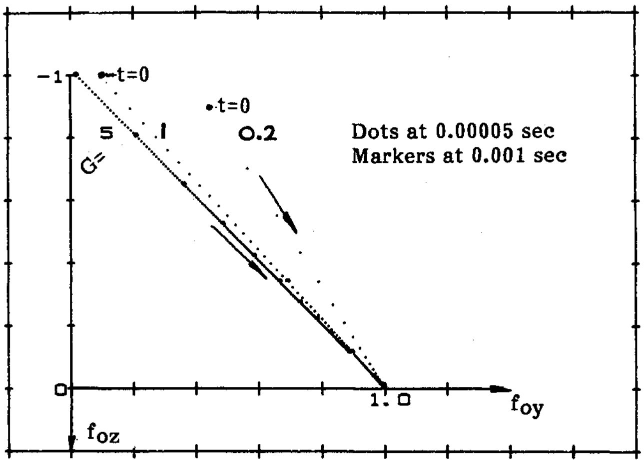

Fig. 8 Response of zero-starred filter (Z^*) to a step demand f_{yd}.

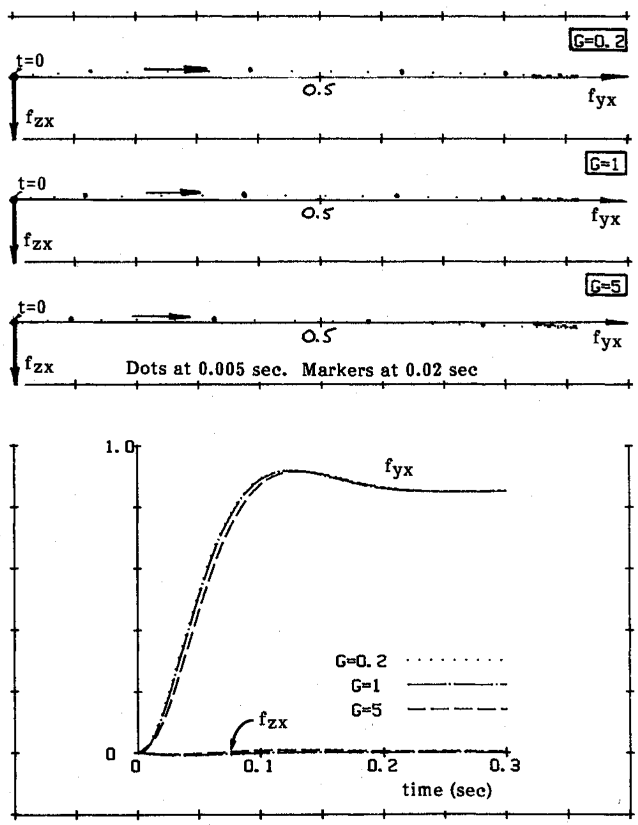

Fig. 9 Missile response with Z^* filter fitted.

following is a practical version of Eq. (39) that has unity gain at both zero and high frequencies

$$F(s) = \frac{1 + s0.9{*}10^{-4}(1 + j10G)}{1 + s0.9{*}10^{-4}\sqrt{1 + 100G^2})} \tag{40}$$

It may be seen that the output of this filter introduces no cross coupling at very low frequencies; it tends to the real number unity as s tends to zero. However, it introduces progressively more as the frequency increases; in fact, as s tends to infinity, then $F(s)$ tends to the complex number $(1 + j10G)/\sqrt{(1 + 100G^2)}$. This indicates that the cross-plane output will be $10G$ times greater than the in-plane one at high frequencies.

The response of the filter to a step input f_{yD} is shown in Fig. 8. The initial output is predominantly cross-plane, but this turns into an in-plane output after an exponential delay that is short when G is low, but which increases as G increases. The filter may be expected to have no particular vices beyond those of its low-pass part, and this may be chosen independently of decoupling considerations. It is this low pass filter that produces the delay at $G = 5$ shown in Fig. 9.

Despite the very different natures of the filters, as may seen by comparing Figs. 6 with 8, they both have the effect of decoupling the maneuver responses. The zero-starred filter (40) leads to the maneuver responses shown in Fig. 9, where the absence of vertical response f_{zx} (compared with that shown in Fig. 5) indicates the success of the technique. Apart from the lag that is caused by the particular low-pass filter chosen, the maneuver responses are not very sensitive to G.

It should be noted that the preceding discussion, in terms of the frequency response, is valid despite the fact that in reality there are two inputs to the complex filter $F(s)$. With the complex summation formulation, both negative and positive frequencies are physically realizable and the designer should be concerned if there is unduly high gain at any frequency. The designer also should be concerned with unobservable or uncontrollable modes in the same way as in a conventional SISO system. One of the advantages of the complex formulation is that most of the conventional design techniques and thinking are readily transferrable to the two-input, two-output situation of the SAC system.

The real equations for the zero-starred filter corresponding to Eqs. (39) will be

$$f_{oz} = (1 + Ts)f_{iz} + s10GTf_{iy} \tag{41a}$$

$$f_{oy} = -s10GTf_{iz} + (1 + Ts)f_{iy} \tag{41b}$$

where $T = 0.9{*}10^{-4}$, and f_0 and f_i are the filter's output and input, respectively.

Complete decoupling in the foregoing example would seem to be unnecessary in view of the smallness of the residue that is left after the prefilter has been added. With the added complexity of inserting complex filters in the feedback paths from the accelerometers and the rate gyroscopes, the complex coefficients in the denominator of the transfer function (36) could be changed into real ones, thereby eliminating all cross coupling.

Conclusions

The complex summation technique is well suited for expressing the dynamic equations of an axisymmetric vehicle when the objective is to decouple its responses in orthogonal planes. The objective becomes one of eliminating all imaginary coefficients in the system's equations.

By proceeding from the differential equations to the transfer-function format, the technique leads naturally to the formulation of filters that eliminate cross coupling. The thought processes involved are a natural extension from those used in designing single-input single-output systems. This provides an alternative technique to using state space and an alternative decoupling solution.

The process has been illustrated with an application to a missile with a flight control system, where the act of rolling introduces momentum bias with consequential gyroscopic coupling. Two different techniques lead to prefilters that are equally effective in their achievement of decoupled maneuver responses:

1) An oscillatory filter that follows from using the method of pole-zero cancellation.

2) A non oscillatory (zero-starred) filter that follows from using the product of complex conjugates, a method that develops naturally from the use of complex summation.

Appendix: Standard Missile

The numerical illustrations in this paper apply to an axisymmetric missile with a length of about 2 m and the following characteristics:

$$Z_w/m = Y_v/m = -1,\ \mathrm{s}^{-1}$$
$$Z_q/m = -Y_r/m = 0,\ \mathrm{m\ s}^{-1}$$
$$Z_{\delta e}/m = -Y_{\delta r}/m = -60,\ \mathrm{m\ s}^{-2}$$
$$M_w/I_y = -N_v/I_z = -0.2,\ \mathrm{m}^{-1}\ \mathrm{s}^{-1}$$
$$M_q/I_y = N_r/I_z = -0.8,\ \mathrm{s}^{-1}$$
$$M_{\delta e}/I_y = N_{\delta r}/I_z = -120,\ \mathrm{s}^{-2}$$
$$U = 600,\ \mathrm{m\ s}^{-1}$$

The abbreviation $MZ_{\delta w}/I_y m = (M_\delta Z_w - M_w Z_\delta)/I_y m = 108$. The FCS data for use with the preceding airframe are the following: the accelerometers' location is at $x = (Z_{\delta e} \cdot I_y)/(M_{\delta e} \cdot m) = 0.5$, m. The accelerometers' gains are $K_f = 0.014$ rad/m s^{-2} (equivalent to about 8 deg/g). The rate gyroscopes' gains are $K_\Omega = 0.333$, rad/rad s^{-1}.

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